A WING WITH THE MAXIMUM LIFT/DRAG RATIO AT SUPERSONIC VELOCITY

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The variational problem for a wing with the maximum lift/drag ratio and given volume is solved with the assumptions that the pressure obeys to Newton's law, that the wing is thin and that the frictional resistance is independent of the form of the profile of the wing.

It is shown that the wing with the maximum lift/drag ratio has a profile that differs from the profile of the wing with minimum wave resistance. The optimum profile depends on the given volume and frictional resistance and may be anything from the wedge, to the profile of the wing with minimum wave resistance.

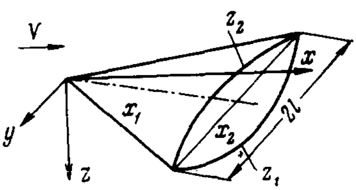

FIG. 1

The possibility of seeking an optimal form for a supersonic aircraft is limited practically by the solution of variational problems formulated with the aid of the simplest relationships between the pressure and the form, for example the Newton's Law, and by the comparison with the bodies of the simplest form, given for example in [1 to 3].

Variational problems, even in the case where Newtonian Law applies, generally reduce to non-linear equations in second order partial derivatives; in order to avoid this, usually the thin body assumption is made. Newtonian Law itself and the aforementioned assumption of a thin body lead to the conclusion that the solution of the variational problem obtained, can be considered only as a guide, furnishing the basis for a more thorough investigation.

In the case of the wing, of course, with these assumptions it is impossible to account properly for the effect of the blunting of the leading edge. Besides, it turns out that the form of the leading edge, with the assumptions used, does not influence the aerodynamic characteristics of the wing, which depend only on the distribution of area over the sweep of the wing (the displacement of the longitudinal sections of the wing in the plane parallel to xz, does not influence the characteristics).

The variational problem of a wing of given volume with minimum wave resistance for a given lift was considered by the author previously, and analogous results were obtained recently in [4]. The solution to the problem of a wing with the maximum lift/drag ratio is given below.

Let the lower and upper surfaces of the wing be given by the functions $z_1 = z_1(x, y)$, $z_2 = z_2(x, y)$ (fig.1). Then, with the aforementioned assumptions the drag, the lift and the volume of the wing may be expressed in the form

$$X = X_0 + \iint (p_1^3 + p_2^3)\, dxdy = X_0 + I_3, \qquad Z = \iint (p_1^2 - p_2^2)\, dxdy = I_2,$$
$$\left(p = \frac{\partial z}{\partial x}\right) \qquad V = \iint (z_1 + z_2)\, dxdy = I_1$$

Here X_0 is the component of the frontal resistance that may be considered approximately independent of the wing profile (frictional resistance, resistance of the blunt edge).

We will designate by the letter K the lift/drag ratio of the wing. The variational problem is posed in the following form:

$$K = \frac{I_2}{I_3 + X_0} = \max \qquad \text{with} \qquad I_1 = \text{const} \tag{1}$$

We shall not vary the contour of the wing in the plane xy. Then, using the fact that the variation $\delta z = (\partial z / \partial \alpha)\, \partial \alpha$, $\delta p = (\partial / \partial x)\, \delta z$, by the condition of equality of variation $\delta (K + \lambda V) = 0$ (λ is an undetermined constant multiplier) we obtain

$$\begin{aligned} &\int \left[\left(2p_1 - \frac{3I_2}{I_3 + X_0} p_1^2\right)\delta z_1 - \left(2p_2 + \frac{3I_2}{I_3 + X_0} p_2^2\right)\delta z_2\right] dy + \\ &+ \iint \left\{\left[\lambda (I_3 + X_0) + \frac{3I_2}{I_3 + X_0} \frac{\partial p_1^2}{dx} - 2\frac{\partial p_1}{\partial x}\right]\delta z_1 + \right. \\ &\left. + \left[\lambda (I_3 + X_0) + \frac{3I_2}{I_3 + X_0} \frac{\partial p_2^2}{\partial x} + 2\frac{\partial p_2}{\partial x}\right]\delta z_2\right\} dxdy = 0 \end{aligned} \tag{2}$$

The first integral is taken along the contour of the wing. From the extremal condition on the functions z_1 and z_2 we obtain

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{3I_2}{I_3 + X_0} p^2{}_1 - 2p_1\right) + \lambda (I_3 + X_0) = 0 \\ &\frac{\partial}{\partial x}\left(\frac{3I_2}{I_3 + X_0} p_2^2 + 2p_2\right) + \lambda (I_3 + X_0) = 0 \end{aligned} \tag{3}$$

Assuming that on the leading edge of the wing z_1 and z_2 are given, that is $\delta z_1 = \delta z_2 = 0$, while the boundary conditions on the trailing edge $x = x_2(y)$ are the expected ones, we obtain for this edge

$$p_1\left(2 - \frac{3I_2}{I_3 + X_0}\; p_1\right) = 0, \qquad p_2\left(2 + \frac{3I_2}{I_3 + X_0}\; p_2\right) = 0 \tag{4}$$

We will restrict ourselves to the wing, on the whole of the surface of which the pressure coefficient is non-negative; then the only possible value for p_2 on the trailing edge becomes $p_2(x_2) = 0$; hence, the value $p_1(x_2) = 0$ proves to be impossible, since, if it were accepted, the equality $p_1(x) = -p_2(x)$, which does not correspond to the conditions of the problem, would follow. Therefore the only value for p_1 is

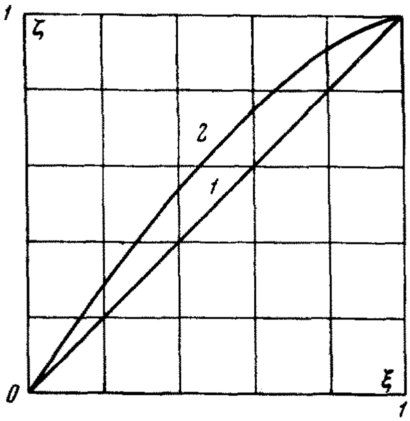

FIG. 2

$$p_1(x_2) = \frac{2}{3}\frac{I_3 + X_0}{I_2}$$

Integrating (3) and determining the arbitrary function of y from the conditions on the trailing edge, we get

$$p_1(x) = \frac{I_3 + X_0}{3I_2}\left[\sqrt{1 + 3I_2\lambda(x_2 - x)} + 1\right], \qquad p_2(x) = \frac{I_3 + X_0}{3I_2}\left[\sqrt{1 + 3I_2\lambda(x_2 - x)} - 1\right] \tag{5}$$

We will continue the integration, with the assumption that the leading edge $x = x_1(y)$ is situated in the plane

$$z = \frac{I_3 + X_0}{3I_2}x$$

Then

$$\begin{aligned} z_1 &= \frac{I_3 + X_0}{3I_2}\left\{\frac{2}{9I_2\lambda}[1 + 3I_2\lambda(x_2 - x_1)]^{3/2} - \frac{2}{9I_2\lambda}[1 + 3I_2\lambda(x_2 - x)]^{3/2} + x\right\} \\ z_2 &= \frac{I_3 + X_0}{3I_2}\left\{\frac{2}{9I_2\lambda}[1 + 3I_2\lambda(x_2 - x_1)]^{3/2} - \frac{2}{9I_2\lambda}[1 + 3I_2\lambda(x_2 - x)]^{3/2} - x\right\} \end{aligned} \tag{6}$$

From the formulas (5) and (6) it may be seen that the resulting profile of the wing is symmetrical with respect to the ordinate

$$z = \frac{I_3 + X_0}{3I_2}\frac{2}{9I_2\lambda}\{[1 + 3I_2\lambda(x_2 - x_1)]^{3/2} - [1 + 3I_2\lambda(x_2 - x)]^{3/2}\} \tag{7}$$

The angle of attack corresponding to the maximum lift/drag ratio is

$$\alpha = \frac{I_3 + X_0}{3I_2} = \frac{1}{3K} \tag{8}$$

The angle of inclination of the profile on the trailing edge equal $\pm\alpha$, and therefore the pressure becomes zero only on the trailing edge of the upper surface of the wing.

The profile of the wing of given volume with minimum drag for the boundary conditions existing on the trailing edge is, of course, symmetrical and is given by the formula

$$z = \lambda[(x_2 - x_1)^{3/2} - (x_2 - x)^{3/2}] \tag{9}$$

Limiting cases for the profile of the wing with the maximum lift/drag ratio are

$$3I_2\lambda(x_2 - x_1) \gg 1$$

$$z \sim [(x_2 - x_1)^{3/2} - (x_2 - x)^{3/2}]$$

$$3I_2\lambda(x_2 - x_1) \ll 1$$

$$z \sim (x - x_1)$$

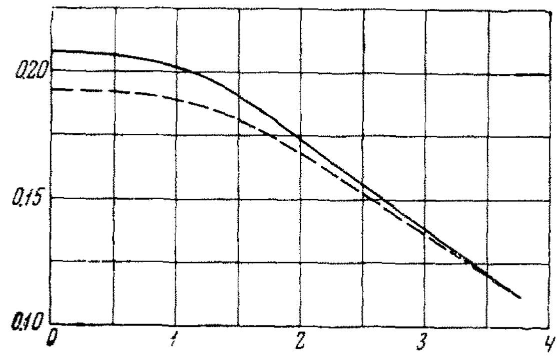

FIG. 3

that is, all the possible profiles of the wing with the maximum lift/drag ratio and given volume are included between the wedge and the profile $1-(1-\xi)^{3/2}$ (fig. 2). For determining the profile (7) we must compute the integrals

$$I_1 = \frac{4(I_3+X_0)}{27I_2^2\lambda}\iint\{[1+3I_2\lambda(x_2-x_1)]^{3/2}-[1+3I_2\lambda(x_2-x)]^{3/2}\}\,dx\,dy$$

$$I_2 = \left(\frac{I_3+X_0}{3I_2}\right)^2\iint\sqrt{1+3I_2\lambda(x_2-x)}\,dx\,dy \tag{10}$$

$$I_3 = 2\left(\frac{I_3+X_0}{3I_2}\right)^3\iint\{[1+3I_2\lambda(x_2-x)]^{3/2}+3\sqrt{1+3I_2\lambda(x_2-x)}\}\,dx\,dy$$

As an example we will give the complete results for the case of a triangular wing, is, $x_2(y)-x_1(y)=k(l-y)$, where l is the half-span of the wing.

$$I_1 = \frac{8l\lambda(I_3+X_0)}{15(3I_2\lambda)^3}\left[(1+3I_2\lambda kl)^{5/2}-\frac{(1+3I_2\lambda kl)^{7/2}-1}{{}^7/_4\,3I_2\lambda kl}+1\right]$$

$$I_2 = \frac{8l}{9I_2\lambda}\left(\frac{I_3+X_0}{3I_2}\right)^2\left[\frac{(1+3I_2\lambda kl)^{5/2}-1}{{}^5/_2\,3I_2\lambda kl}-1\right] \tag{11}$$

$$I_3 = \frac{4l}{15I_2\lambda}\left(\frac{I_3+X_0}{3I_2}\right)^3\left[\frac{(1+3I_2\lambda kl)^{7/2}-1}{{}^7/_2\,3I_2\lambda kl}+\frac{(1+3I_2\lambda kl)^{5/2}-1}{{}^1/_2\,3I_2\lambda kl}-6\right]$$

At this stage we shall introduce dimensionless quantities: $i_2 = 3I_2\lambda kl$, $i_3 = I_3/l^2$, $x_0 = X_0/l^2$, $i_1 = I_1/l^3k^{5/3}$, $m = \lambda l^3k^{4/3}$. Substituting these into (11) we obtain

$$8(x_0+i_3)^2m^3 = \frac{{}^5/_2\,i_2^5}{(1+i_2)^{5/2}-{}^5/_2\,i_2-1}$$

$$\frac{5i_3}{4(x_0+i_3)^3m^3} = \frac{(1+i_2)^{7/2}+7(1+i_2)^{5/2}-21i_2-8}{{}^7/_2\,i_2^5} \tag{12}$$

$$\frac{15i_1}{8(x_0+i_3)m} = \frac{1}{i_2^3}\left[1+(1+i_2)^{5/2}-\frac{(1+i_2)^{7/2}-1}{{}^7/_4\,i_2}\right]$$

With the help of the formulas (12) it is not difficult to construct graphs showing the dependence of i_2, i_3, and m on the given parameters x_0 and i_1.

For the wing with the wedge profile

$$Kv = \frac{1}{\sqrt{27 + v^{-3}\xi}}, \quad v = \frac{V}{k^2 l^3}, \quad \xi = \frac{X_0}{kl^2} \tag{13}$$

The curve Kv, as a function of $v^{-1}\xi^{1/3}$ is given in fig. 3 (dotted line) for comparison with the wing with the maximum lift/drag ratio. For values of the parameter $v^{-1}\xi^{1/3}$ greater than those given in fig. 3, the optimum profile, apparently, must include a part on which $p_2 = 0$.

From the formula (13) it may be seen that the optimum dimension of the wing is

$$v = {}^1/_3 \left(\frac{\xi}{2}\right)^{1/3}, \qquad K = {}^1/_3 \left(\frac{2}{\xi}\right)^{1/3}$$

BIBLIOGRAPHY

1. Maikapar G.I. O volnovom soprotivlenii neosesimmetrichnykh tel v sverkhzvukovom potoke (On the wave resistance of a non-axisymmetric body in a supersonic stream). *PMM*, Vol. 23, No. 2, 1959.

2. Maikapar G.I. O postroenii sverkhzvukovogo techeniia obtekaniia tverdykh tel pri pomoshchi ploskikh skachkov uplotneniia (On the construction of a supersonic flow around a streamlined solid body using planar density jumps). Trans. Acad. Sci. USSR Mech. and Mech. Engr., No. 5, 1964.

3. Maikapar G.I. Tela, obrazovannye poverkhnostiami toka konicheskikh techenii (Bodies generated by stream surfaces in conical flows). Mekhanika zhidkosti i gaza. (Mech. Liquids and Gases) No. 1, 1966.

4. Hull D., Miele A. Three dimensional wings of minimum total drag in Newtonian flow. J. Astron. Sci., Vol. 12, No. 2, 1965.

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